NASA TECHNICAL MEMORANDUM

NASA TM X-53079

OCTOBER 7, 1964

NASA TM X-53079

N65 13575

(ACCESSION NUMBER) (THRU)

24 1

(PAGES) (CODE)

TMX-53079 19

(NASA CR OR TMX OR AD NUMBER) (CATEGORY)

FACILITY FORM 602

AN EXPLICIT SOLUTION FOR LARGE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

by ROBERT E. CUMMINGS AND LYLE R. DICKEY
Aero-Astrodynamics Laboratory

NASA

George C. Marshall Space Flight Center, Huntsville, Alabama

GPO PRICE $ ______

OTS PRICE(S) $ ______

Hard copy (HC) 1.00

Microfiche (MF) .50

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AN EXPLICIT SOLUTION FOR LARGE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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Robert E. Cummings and Lyle R. Dickey

George C. Marshall Space Flight Center

Huntsville, Alabama

ABSTRACT

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A practical method of obtaining an explicit solution to an inhomogeneous system of linear differential equations with constant coefficients is described. The method is readily adaptable to solving large systems on high speed digital computers and is particularly efficient when a large number of solutions are desired for the same set of equations with different initial conditions and forcing functions. The problem that often arises when large eigenvalues are present is overcome by a unique feature. The solution is obtained for an exceptionally small integration step and a process is described whereby the step can be doubled. Successive applications of this process provide a solution over an interval which increases exponentially in size with each step whereas the work involved increases only in a linear fashion. This is particularly advantageous since standard techniques require that special provisions must be made for any system which has exceptionally large eigenvalues.

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SUMMARY

A practical method of obtaining an explicit solution to an inhomogeneous system of linear differential equations with constant coefficients is described. The method is readily adaptable to solving large systems of equations containing large eigenvalues. The solution is first obtained at a small interval. A process is described which yields the solution at twice this interval by several matrix operations. Successive applications of this operation increase the interval exponentially while the number of operations required increases linearly. It is feasible to choose an exceptionally small interval initially and thus avoid the usual problems associated with large eigenvalues. The solution is expressed explicitly as a function of initial conditions and unspecified parameters for a large class of forcing functions.

I. INTRODUCTION

A great amount of effort and computer time is frequently required to determine solutions to a given set of differential equations for a large number of different initial conditions and forcing functions. Of these differential equations, a large number are or can be represented by a set of linear differential equations. For a rather large class of forcing functions, these equations can be solved explicitly as a linear combination of initial conditions and forcing function parameters. A specific solution is then reduced to evaluating the inner product of two vectors for any given variable at a prespecified time. Such an explicit solution permits rapid evaluation of large statistical samples of initial conditions and forcing functions in addition to providing considerable additional insight into the problem. As a first step in this direction, this paper presents a solution for the special case where the coefficients of the system are constant.

II. GENERAL SOLUTION FOR CONSTANT COEFFICIENTS

In matrix notation, a constant coefficient, linear system of differential equations can always be expressed in the following form:

$$\dot{X} = AX + F(t) \tag{1}$$

where X, $\dot{X}$ and F(t) are vectors and A is a matrix of constant coefficients.

This equation has the following solution (Reference 1, p. 169):

$$X(t) = e^{A(t-t_o)} X(t_o) + \int_{t_o}^{t} e^{A(t-\tau)} F(\tau)\, d\tau, \tag{2}$$

where

$$e^{A(t-\tau)} = \sum_{n=0}^{\infty} \frac{A^n (t - \tau)^n}{n!} . \tag{3}$$

The problem of solving the system defined by equation (1) is thus reduced to evaluating this series and performing the integration indicated in equation (2).

III. EXPLICIT DECOMPOSITION OF THE FORCING FUNCTION

To evaluate the integral of equation (2) explicitly as a linear combination of parameters which characterize the forcing function, we may, with no loss of generality, write

$$F(t) = \sum_{i=0}^{p} F_i \, f_i(t), \tag{4}$$

where the F_i are constant vectors and the $f_i(t)$ are scalar variables. To proceed further toward an explicit solution, consider

$$\int_{t-s}^{t} e^{A(t-\tau)} F_i \, f_i(\tau) \, d\tau.$$

It will be assumed that, over the interval s, the scalar variable $f_i(t)$ is an element of the set of all functions which can be written as a linear combination of $m + 1$ basis functions which are to be specified. The coefficients of the basis functions are the parameters of the set of forcing functions and are to be left arbitrary for present purposes. Thus, we may write

$$f_i(t) = G(t) \, B_i, \tag{5}$$

where

$$G(t) = \Big(g_m(t), \; g_{m-1}(t), \; \ldots, \; g_o(t)\Big)$$

is a vector whose components are the basis functions and

$$B_i = \begin{bmatrix} b_m \\ b_{m-1} \\ \cdot \\ \cdot \\ \cdot \\ b_o \end{bmatrix}$$

is a vector whose components are parameters which are independent of the variable of integration. Thus, we have

$$\int_{t-s}^{t} e^{A(t-\tau)} F_i \, f_i(\tau) \, d\tau = \left[\int_{t-s}^{t} e^{A(t-\tau)} F_i \, G(\tau) \, d\tau \right] B_i. \tag{6}$$

This decomposition of the forcing functions allows the necessary integration to be performed independent of the parameters of the forcing function and, therefore, yields the result explicitly as a function of these parameters.

IV. EXPONENTIAL EXPANSION OF THE INTEGRATION INTERVAL

The series in equation (3) can be effectively evaluated by taking advantage of the important property,

$$e^{At_1} e^{At_2} = e^{A(t_1+t_2)}, \tag{7}$$

for which a proof appears in Reference 1.

In particular, it may be noted that if $t_1 = t_2 = \delta t$,

$$\left[e^{A\delta t}\right]^2 = e^{A(2\delta t)}.$$

Thus, if the series in equation (3) is evaluated at δt, its value at $2\delta t$ can be determined by simply squaring the matrix obtained for δt. This can be extended to $2^k\ \delta t$ by k matrix multiplications.

The integral of equation (6) over an interval $2\delta t$ can be expressed as follows for each value of i:

$$\left[\int_{t-2\delta t}^{t} e^{A(t-\tau)}\ F_i\ G(\tau)\ d\tau\right] B_i.$$

This integral can then be expressed as a sum of two integrals.

$$\int_{t-2\delta t}^{t} e^{A(t-\tau)} F_i G(\tau)\, d\tau = \int_{t-2\delta t}^{t-\delta t} e^{A(t-\tau)} F_i G(\tau)\, d\tau$$

$$+ \int_{t-\delta t}^{t} e^{A(t-\tau)} F_i G(\tau)\, d\tau.$$

A transformation on the variable of integration of the first of these integrals yields

$$\int_{t-2\delta t}^{t-\delta t} e^{A(t-\tau)} F_i G(\tau)\, d\tau = \int_{t-\delta t}^{t} e^{A(t-\tau+\delta t)} F_i G(\tau - \delta t)\, d\tau$$

$$+ \int_{t-\delta t}^{t} e^{A(t-\tau)} F_i G(\tau)\, d\tau.$$

The property described in equation (7) implies that

$$e^{A(t-\tau+\delta t)} = e^{A\delta t} e^{A(t-\tau)}.$$

Furthermore, it will be required that the basis functions $G(t)$ have the following property:

$$G(\tau - \delta t) = G(\tau)\, T(- \delta t). \tag{8}$$

Then,

$$\int_{t-2\delta t}^{t} e^{A(t-\tau)} F_i \, G(\tau)\, d\tau = e^{A\delta t} \left[\int_{t-\delta t}^{t} e^{A(t-\tau)} F_i \, G(\tau)\, d\tau \right] T(-\delta t)$$

$$+ \left[\int_{t-\delta t}^{t} e^{A(t-\tau)} F_i \, G(\tau)\, d\tau \right] . \tag{9}$$

This result shows that the integral over $2\delta t$ can be determined by two matrix multiplications and one matrix addition provided only that the integral be known over δt, that $e^{A\delta t}$ is known and that $T(-\delta t)$ is defined. Choosing $f_i(t)$ to be polynomials provides the following definitions:

$$g_j(t) = t^j, \quad j = 0, 1, \ldots, m.$$

$$T(\delta t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \binom{m}{1}\delta t & 1 & 0 & \cdots & 0 \\ \binom{m}{2}\delta t^2 & \binom{m-1}{1}\delta t & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \binom{m}{m}\delta t^m & \binom{m-1}{m-1}\delta t^{m-1} & \binom{m-2}{m-2}\delta t^{m-2} & \cdots & 1 \end{bmatrix}$$

In order to evaluate the integral required in equation (9), we get from equation (6) that

$$\int_{t-\delta t}^{t} e^{A(t-\tau)} F_i f_i(\tau)\, d\tau = \int_{t-\delta t}^{t} e^{A(t-\tau)} F_i G(\tau)\, d\tau\, B_i,\ 0 \leqq \delta t \leqq s.$$

Let $\tau' = t - \tau$ be a change of variable of integration. Then

$$\int_{t-\delta t}^{t} e^{A(t-\tau)} F_i G(\tau)\, d\tau\, B_i = \int_{0}^{\delta t} e^{A\tau'} F_i G(t - \tau')\, d\tau'\, B_i$$

$$= \int_{0}^{\delta t} e^{A\tau'} F_i G(-\tau')\, d\tau'\, T(t)\, B_i$$

by using equation (8). Now defining

$$I^* = \begin{bmatrix} (-1)^m & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & -1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix},$$

i.e., $I^*_{ij} = 0, \quad i \neq j$

$$= (-1)^{m+1-i},\ i = j$$

gives

$$G(-\tau') = G(\tau') \, I^*.$$

Thus, we have

$$\int_{t-\delta t}^{t} e^{A(t-\tau)} \, F_i \, G(\tau) \, B_i = \int_{o}^{\delta t} e^{A\tau'} \, F_i G(\tau') \, d\tau' \, I^* \, T(t) \, B_i$$

$$= P_i(\delta t) \, I^* \, T(t) \, B_i$$

where

$$P_i(\delta t) = \int_{o}^{\delta t} e^{A\tau} \, F_i G(\tau) \, d\tau. \tag{10}$$

Bellman [1] proved that the series of equation (3) is uniformly convergent in any finite interval $(0, t - \tau)$. Furthermore, a sufficiently small interval $(0, \delta t)$, where $0 < \delta t \leqq s$ can be specified such that the matrix series defining $e^{A\delta t}$ and $P_i(\delta t)$ can be evaluated with negligible error by a small number of terms.

With $P_i(\delta t)$ evaluated we have

$$\int_{t-\delta t}^{t} e^{A(t-\tau)} \, F_i \, G(\tau) \, d\tau \, B_i = P_i(\delta t) \, I^* \, T(t) \, B_i. \tag{11}$$

Substituting this result into equation (9) gives

$$\int_{t-2\delta t}^{t} e^{A(t-\tau)} F_i G(\tau)\, d\tau\, B_i = \left\{ e^{A\delta t} P_i(\delta t)\, I^* T(t)\, T(-\delta t) + P_i(\delta t)\, I^* T(t) \right\} B_i .$$

This may be simplified by proving

$$I^*\, T(t)\, T(-\delta t) = T(\delta t)\, I^* T(t). \qquad (12)$$

From equation (8) we get

$$G[\tau + (\tau_1 + \tau_2)] = G(\tau)\, T(\tau_1 + \tau_2)$$

$$G[(\tau + \tau_1) + \tau_2] = G(\tau + \tau_1)\, T(\tau_2) = G(\tau)\, T(\tau_1)\, T(\tau_2).$$

$$G[(\tau + \tau_2) + \tau_1] = G(\tau + \tau_2)\, T(\tau_1) = G(\tau)\, T(\tau_2)\, T(\tau_1).$$

$$\therefore\ T(\tau_1 + \tau_2) = T(\tau_1)\, T(\tau_2) = T(\tau_2)\, T(\tau_1).$$

Also by definition

$$G(-\tau') = G(\tau')\, I^*$$

for any τ'. Thus,

$$G(-\tau + \tau_1) = G(-\tau)\, T(\tau_1) = G(\tau)\, I^* T(\tau_1)$$

but also

$$G(-\tau + \tau_1) = G(\tau - \tau_1)\, I^* = G(\tau)\, T(-\tau_1)\, I^*.$$

Therefore,

$$I^*T(\tau_1) = T(-\tau_1)\ I^*.$$

Thus,

$$I^*T(t)\ T(-\delta t) = I^*T(-\delta t)\ T(t) = T(\delta t)\ I^*T(t)$$

which establishes equation (12).

Using this gives

$$\int_{t-2\delta t}^{t} e^{A(t-\tau)}\ F_i G(\tau)\ d\tau\ B_i = \left\{ e^{A\delta t}\ P_i(\delta t)\ T(\delta t) + P_i(\delta t) \right\} I^*T(t)\ B_i) \tag{13}$$

provided only that $0 < 2\delta t \leqq s$.

Thus, given the integral by equation (11) over an interval δt, equation (13) shows that the interval can be doubled by two matrix multiplications and one addition and requiring only $e^{A\delta t}$, $P_i(\delta t)$, and $T(\delta t)$ all of which are available.

Thus, the integral of equation (2) can be evaluated explicitly as a linear combination of the coefficients of an arbitrary polynomial of degree less than or equal to m. This can be accomplished for a point $t_o + \triangle t$ (where $\triangle t = 2^k\ \delta t \leqq s$) by 4k matrix multiplications and k matrix additions. This results in the integration interval being expanded exponentially while the computer time increases only linearly. This exponential expansion of the integration interval then gives us

$$\int_{t-\triangle t}^{t} e^{A(t-\tau)}\ F_i G(\tau)\ d\tau\ B_i = P_i(\triangle t)\ I^*T(t)\ B_i. \tag{14}$$

V. SOLUTION AT EVENLY SPACED POINTS

Given the initial conditions at time t_o, the above results can be employed to efficiently obtain the explicit solution at later times. From equations (2), (4), and (5) we have

$$X(t_o + \Delta t) = e^{A\Delta t} X(t_o) + \sum_i \int_{t_o}^{t_o+\Delta t} e^{A(t-\tau)} F_i G(\tau) \, d\tau \, B_i,$$

by equation (14),

$$X(t_o + \Delta t) = e^{A\Delta t} X(t_o) + \sum_i P_i(\Delta t) \; I{*}T(t_o + \Delta t) \; B_i,$$

and by equation (12),

$$X(t_o + \Delta t) = e^{A\Delta t} X(t_o) + \sum_i P_i(\Delta t) T(-\Delta t) \; I{*}T(t_o) \; B_i. \tag{15}$$

Defining

$$\bar{B}_{io} = I{*}T(t_o) \; B_i,$$

we have

$$X(t_o + \Delta t) = e^{A\Delta t} X(t_o) + \sum_i P_i(\Delta t) \; T(-\Delta t) \; \bar{B}_{io}.$$

Now defining

$$m_i(\Delta t) = P_i(\Delta t)\ T(-\Delta t),$$

gives

$$X(t_o + \Delta t) = e^{A\Delta t}\ X(t_o) + \sum_i m_i(\Delta t)\ \bar{B}_{io}. \tag{16}$$

Thus, we have the response at $t = t_o + \Delta t$ as a linear combination of initial conditions at $t = t_o$ and the parameters $\bar{B}_{io}$ which characterize the forcing functions in the interval $t_o \leqq t \leqq t_o + \Delta t$.

More generally we get from equation (15) for $k\Delta t \leq s$,

$$X(t_o + k\Delta t) = e^{Ak\Delta t}\ X(t_o) + \sum_i P_i(k\Delta t)\ T(-k\Delta t)\ I{*}T(t_o)\ B_i$$

$$X(t_o + k\Delta t) = e^{Ak\Delta t}\ X(t_o) + \sum_i m_i(k\Delta t)\ \bar{B}_{io}, \quad k = 1, 2, 3, \ldots. \tag{17}$$

However, by writing $t_o + k\Delta t = t_o + (k - 1)\ \Delta t + \Delta t$, we get similarly from equations (14) and (15)

$$X(t_o + k\Delta t) = e^{A\Delta t}\ X[t_o + (k - 1)\ \Delta t] + \sum_i P_i(\Delta t)\ I{*}T(t_o + k\Delta t)\ B_i$$

$$X(t_o + k\Delta t) = e^{A\Delta t}\ X[t_o + (k - 1)\ \Delta t] + \sum_i m_i(\Delta t)\ T[-\ (k - 1)\ \Delta t]\ I{*}T(t_o)\ B_i$$

$$X(t_o + k\Delta t) = e^{A\Delta t}\ X[t_o + (k - 1)\ \Delta t] + \sum_i m_i(\Delta t)\ T[-\ (k - 1)\ \Delta t]\ \bar{B}_{io}. \tag{18}$$

But by equation (17)

$$X[t_o + (k - 1)\,\Delta t] = e^{A(k-1)\Delta t}\, X(t_o) + \sum_i m_i\, [(k - 1)\Delta t]\, \bar{B}_{io}$$

which with equation (18) and equation (7) gives

$$X(t_o + k\Delta t) = e^{Ak\Delta t}\, X(t_o) + \sum_i e^{A\Delta t}\, m_i\, [(k - 1)\,\Delta t]\, \bar{B}_{io}$$

$$+ \sum_i m_i(\Delta t)\, T[-(k - 1)\Delta t]\, \bar{B}_{io}$$

$$= e^{Ak\Delta t}\, X(t_o) + \sum_i \left\{ e^{A\Delta t}\, m_i[(k - 1)\Delta t] + m_i(\Delta t)\, T[-(k-1)\Delta t] \right\} \bar{B}_{io}.$$

Comparing this result with equation (17) shows

$$m_i(k\Delta t) = e^{A\Delta t}\, m_i[(k - 1)\Delta t] + m_i(\Delta t)\, T[-(k - 1)\Delta t] \tag{19}$$

where

$$m_i(\Delta t)\, T[-(k - 1)\Delta t] = \left\{ m_i(\Delta t)\, T[-(k - 2)\Delta t] \right\} T(-\Delta t),\ k = 1, 2, \ldots.$$

Thus, equation (17) gives the solution at successive evenly spaced points by using the recursive relation of equation (19). This requires only $e^{A\Delta t}$, $m_i(\Delta t)$ and $T(-\Delta t)$ all of which have been previously determined.

Equation (17) gives the solution at any point $t_o + k\triangle t$ for which $k \leqq k_1$, where k_1 is the largest value of k for which the forcing functions are described by the same vector B_i of equation (5).

If more generally we have

$$f_i(t) = G(t)\ B_{io} \qquad 0 \leqq t \leqq t_o + k_1\triangle t = t_{k_1} \tag{20}$$

$$f_i(t) = G(t)\ B_{i1} \qquad t_{k_1} \leqq t \leqq t_{k_1} + k_1\triangle t$$

then by equation (17)

$$X(t_o + k\triangle t) = e^{Ak\triangle t}\ X(t_o) + \sum_i m_i(k\triangle t)\ \bar{\bar{B}}_{io}, \quad k = 1, 2, \ldots k_1$$

$$X(t_{k_1} + k\triangle t) = e^{Ak\triangle t}\ X(t_{k_1}) + \sum_i m_i(k\triangle t)\ \bar{\bar{B}}_{i1}, \quad k = 1, 2, \ldots k_1$$

where

$$\bar{\bar{B}}_{i_1} = I^*T(t_{k_1})\ B_{i1}. \tag{21}$$

Thus

$$X(t_{k_1} + k\triangle t) = e^{A(k_1+k)\triangle t}\ X(t_o) + \sum_i e^{Ak\triangle t}\ m_i(k_1\triangle t)\ \bar{\bar{B}}_{io}$$

$$+ \sum_i m_i(k\triangle t)\ \bar{\bar{B}}_{i1}, \qquad k = 1, 2, \ldots k_1. \tag{22}$$

By using the recursive relations of equation (19) we get the solution at successive evenly spaced points explicitly as a linear combination of initial conditions and the vectors $\bar{B}_{io}$ and $\bar{B}_{i1}$ which describe the forcing functions.

VI. INTERPOLATING POLYNOMIALS

The use of polynomials which are defined differently over different intervals adds considerably to the number of parameters that appear in the solution. For simplicity of notation it will be assumed that the same polynomial for $f_i(t)$ applies throughout an interval $k_1\Delta t$ but differs from one such interval to the next. The change in definition of the polynomial for $f_i(t)$ then is at a point at which a solution is obtained. Furthermore, it will be assumed that in each adjoining interval of length $k_1\Delta t$ each forcing function $f_i(t)$ is represented by an interpolation scheme whereby the coefficients of the polynomials are a linear combination of several discrete values of $f_i(t)$. Let f_i^* be the vector of all the discrete values of $f_i(t)$ to be used. Generalizing equation (20) gives

$$f_i(t) = G(t)\ B_{ij}, \qquad t_o + jk_1\Delta t \leqq t \leqq t_o + (j + 1)\ k_1\Delta t \tag{23}$$

where the vector B_{ij} which describes the interpolation scheme is now given by

$$B_{ij} = B_{ij}^*\ f_i^* \tag{24}$$

where B_{ij}^* is a matrix of constants defining the interpolation scheme. Similarly, from equation (21), we generalize to get

$$\bar{B}_{ij} = I^*T(t_o + jk_1\Delta t)\ B_{ij}. \tag{25}$$

Thus, from equation (24)

$$\bar{B}_{ij} = I^*T(t_o + jk_1\Delta t)\ B_{ij}^*\ f_i^*. \tag{26}$$

Now defining

$$\bar{B}^*_{ij} = I^*T(t_o + jk_1\triangle t)\ B^*_{ij} \tag{27}$$

gives

$$\bar{B}_{ij} = \bar{B}^*_{ij}\ f^*_i. \tag{28}$$

Thus, from equation (28) and equation (17), we get

$$X(t_o + k\triangle t) = e^{Ak\triangle t}\ X(t_o) + \sum_i m_i(k\triangle t)\ \bar{B}^*_{io}\ f^*_i, \quad \text{for } k = 1, 2, \ldots k_1. \tag{29}$$

From equation (22), we get

$$\begin{aligned} X[t_o + (k_1 + k)\triangle t] &= e^{A(k_1+k)\triangle t}\ X(t_o) + \sum_i \Big\{ e^{Ak\triangle t}\ m_i(k_1\triangle t)\ \bar{B}^*_{io} \\ &\quad + m_i(k\triangle t)\ \bar{B}^*_{i1} \Big\}\ f^*_i \\ &= e^{A(k_1+k)\triangle t}\ X(t_o) + \sum_i M^*_i[(k_1+k)\triangle t] f^*_i \end{aligned} \tag{30}$$

where

$$M^*_i[(k_1+k)\triangle t] = e^{Ak\triangle t}\ m_i(k_1\triangle t)\ \bar{B}^*_{io} + m_i(k\triangle t)\ \bar{B}^*_{i1}. \tag{31}$$

Continuing this process, we get

$$X[t_o + (jk_1 + k)\triangle t] = e^{A(jk_1+k)\triangle t}\ X(t_o) + \sum_i M^*_i[(jk_1 + k)\triangle t]\ f^*_i \tag{32}$$

where

$$M_i^*[(jk_1 + k)\Delta t] = e^{Ak\Delta t} M_i^*(jk_1\Delta t) + m_i(k\Delta t)\, \bar{B}_{ij}^*, \qquad k = 1, 2, \ldots k_1$$
$$j = 0, 1, 2, \ldots \tag{33}$$

and

$$M_i^*(0) = \mathbb{O},$$

the null matrix. Thus, we have in general

$$X(t_o + \eta\Delta t) = e^{A\eta\Delta t} X(t_o) + \sum_i M_i^*(\eta\Delta t)\, f_i^*, \qquad \eta = 1, 2, \ldots N \tag{34}$$

where f_i^* is the vector of all the discrete values of $f_i(t)$ required by the interpolation scheme to represent the forcing functions over the range t_o to t_N where t_N is the last point for which a solution is required. This is the desired general explicit solution.

VII. CONCLUSIONS

The method just described is particularly applicable to statistical studies requiring a large number of solutions to the same differential equations with different initial conditions and forcing functions. It should be noticed, however, that this is not the only advantage. Of particular interest is the manner in which the size of the integration step is increased. Whereas most techniques increase the integration step in direct proportion to the work required, this technique increases the step size exponentially while the number of operations required increases linearly. This is extremely important in systems that contain large eigenvalues. In many such cases, explicit solutions can be obtained where the time required by standard techniques would practically prohibit a solution. The employment of this technique can be used to reduce computer time required for a large number of solutions or tremendously increase the number of solutions that can be obtained in the same time.

In addition to the actual solutions obtained, the explicit solution itself provides considerable insight. Essentially it gives all the trade-off factors simultaneously and the effect of changes in the value of the initial condition or forcing function is apparent and immediately available from this solution.

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1. Bellman, Richard, "Introduction to Matrix Analysis," McGraw Hill, New York, 1960, Chapter 10.

October 7, 1964 APPROVAL NASA TM X-53079

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By Robert E. Cummings and Lyle R. Dickey

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Lockheed Research and Engineering Center
Huntsville, Alabama
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Mr. Ray Bieber
Mr. Robert Chapman
Mr. G. H. Parker
Dr. Charlotte Striebel

Dr. Herbert A. Meyer
Dept. of Statistics
University of Fla.
Gainesville, Fla.

Dr. H. O. Hartley
The Institute of Statistics
Texas A&M University
College Station, Texas

Dr. Richard Bellman
The Rand Corporation
Santa Monica, California

Mr. James E. Pope
Chrysler Corporation
New Orleans, Michoud

Tech. & Sci. Info. Facility (25)
P. O. Box 5700
Bethesda, Md.
Attn: NASA Rep. (S-AK/RKT)

Mr. Gary A. Kinemond
Div. 9130
Sandia Corp.
P. O. Box 5800
Albuquerque, New Mexico